Some generalizations of the almost periodic functions

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(joint work with S. Pilipović, M. Kostić, M. T. Khalladi and A. Rahmani)

References

1. E. Alvarez, S. Castillo, M. Pinto, (ω, c)-*Pseudo periodic functions, first order Cauchy problem and Lasota-Wazewska model with ergodic and unbounded oscillating production of red cells,* Bound. Value Probl. Article No. 106, (2019), 1–20.
2. M. F. Hasler, G. M. N'. Guérékata, *Bloch-periodic functions and some applications,* Nonlinear Studies **21**, (2014), 21–30.
3. M. Kostić, *Almost Periodic and Almost Automorphic Type Solutions of Abstract Volterra Integro–Differential Equations,* De Gruyter, Berlin/Boston, 2019.
4. M. T. Khalladi, M. Kostić, A. Rahmani, D. Velinov, (ω, c)-*pseudo almost periodic functions,* (ω, c)-*pseudo almost automorphic and applications,* Facta Universitatis, accepted.

References

5. M. T. Khalladi, M. Kostić, A. Rahmani, D. Velinov *(ω, c)-almost periodic type functions and applications,* submitted.
6. B. Chaouchi, M. Kostić, S. Pilipović, D. Velinov, *Semi-Bloch periodic functions, semi-anti-periodic functions and applications,* Chelyabinsk Phys. and Math. J. **5**, (2020), 243–255.
7. J. Andres, D. Pennequin, *Semi-periodic solutions of difference and differential equations,* Bound. Value Probl. **141**, (2012), 1–16.
8. M. T. Khalladi, M. Kostić, M. Pinto, A. Rahmani, D. Velinov, *c-almost periodic type functions and applications,* submitted.

Plan of the talk

1. Introduction
2. Semi-Bloch k-periodic functions
3. (ω, c)-almost periodic functions
   - (ω, c)-periodic functions
   - (ω, c)-almost periodic functions
   - Composition principles, convolution products
4. Applications

Heat equation

The unique solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, with the initial condition $u(x,0) = f(x)$ is given by

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\, ds, \quad x \in \mathbb{R}, \ \ t \geq 0.$$

Mathieu linear differential equation

$$y''(t) + \Big(a - 2qcost\Big) y(t) = 0.$$

Almost periodic functions

We put $I = \mathbb{R}$ or $I = [0, \infty)$, and $f : I \to X$ be continuous function. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ iff

$$\|f(t+\tau) - f(t)\| \leq \epsilon, \quad t \in I.$$

The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic, (AP) for short, if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $r > 0$ such that any subinterval of I of length r meets $\vartheta(f, \epsilon)$. The vector space consisting of all almost periodic functions is denoted by $AP(I : X)$.

Almost automorphic functions

Let $f : \mathbb{R} \to E$ be continuous. It is said that $f(\cdot)$ is almost automorphic if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to E$ such that

$$\lim_{n\to\infty} f(t + a_n) = g(t) \qquad \text{and} \qquad \lim_{n\to\infty} g(t - a_n) = f(t),$$

pointwise for $t \in \mathbb{R}$.

Stepanov p-almost periodic functions

Let $1 \leq p < \infty$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Stepanov p-bounded, S^p-bounded shortly, if

$$\|f\|_{S^p} := \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

A function $f \in L^p_S(\mathbb{R} : X)$ is said to be Stepanov p-almost periodic, (S^p-AP) shortly, if the function $\hat{f} : \mathbb{R} \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in \mathbb{R}, \ s \in [0,1]$$

is almost periodic.

Uniformly recurrent functions, A. Haraux, P. Souplet

We say that the function $f(\cdot)$ is uniformly recurrent if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|f(t+\alpha_n) - f(t)\| = 0.$$

Bloch (p, k)-periodic functions

Let $I = [0, \infty)$ or $I = \mathbb{R}$, $p \geq 0$ and $k \in \mathbb{R}$. A bounded continuous function $f : I \to E$ is said to be Bloch (p, k)-periodic, or Bloch periodic with period p and Bloch wave vector or Floquet exponent k if

$$f(x + p) = e^{ikp} f(x), \qquad x \in I.$$

Antiperiodic functions

For the function $f : I \to E$ it is said that is antiperiodic if there exists $p > 0$ such that $f(x + p) = -f(x)$ for all $x \in I$.

Semi-Bloch k-periodic functions

For a bounded continuous function $f \in \mathcal{C}(\mathbb{R} : E)$ is said to be a semi-Bloch k-periodic if

$$\forall \varepsilon > 0 \quad \exists p \geq 0 \quad \forall m \in \mathbb{Z} \quad \forall x \in \mathbb{R} \quad \|f(x + mp) - e^{ikp} f(x)\| \leq \varepsilon.$$

Semi-anti-periodic functions

For a bounded continuous function $f : I \to E$ it is said that is semi-anti-periodic if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{Z} \quad \forall x \in I \quad \|f(x + mp) - (-1)^m f(x)\| \leq \varepsilon.$$

Theorem

Let $k \in \mathbb{R}$ and $f : I \to E$ is bounded continuous function. Then the following holds:

i) $f(\cdot)$ is semi-Bloch k-periodic iff $e^{-ik\cdot} f(\cdot)$ is semi-periodic;

ii) $f(\cdot)$ is semi-Bloch k-periodic iff there exists a sequence of continuous periodic functions (f_n), $f_n : I \to E$, for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} e^{ikx} f_n(x) = f(x)$ uniformly in I;

iii) $f(\cdot)$ is semi-Bloch k-periodic iff there exists a sequence of Bloch k-periodic functions (f_n), $f_n : I \to E$, for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in I.

Example

The function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{ix}/(2n+1)}{n^2}, \quad x \in \mathbb{R}$$

is semi-anti-periodic because it is uniform limit of $[\pi \cdot (2n+1)!!]$-anti-periodic functions

$$f_N(x) = \sum_{n=1}^{N} \frac{e^{ix}/(2n+1)}{n^2}, \quad x \in \mathbb{R} \ (N \in \mathbb{N}).$$

Theorem

Let $k \in \mathbb{R}$, $p > 0$ and let $f : I \to X$. Then we have:

i) If $f(\cdot)$ is Bloch (p, k)-periodic (semi-Bloch k-periodic, semi-anti-periodic), then $cf(\cdot)$ is Bloch (p, k)-periodic (semi-Bloch k-periodic, semi-anti-periodic) for any $c \in \mathbb{C}$;

ii) If $X = \mathbb{C}$, $\inf_{x \in \mathbb{R}} |f(x)| = m > 0$ an $f(\cdot)$ is Bloch (p, k)-periodic (semi-Bloch k-periodic, semi-anti-periodic), then $1/f(\cdot)$ is Bloch $(p, -k)$-periodic (semi-Bloch $(-k)$-periodic, semi-anti-periodic).

Example

The solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, with initial condition $u(x,0) = f(x)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic) if the function $f(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic).

(ω, c)-periodic functions

Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. A continuous function $f : I \to E$ is said to be (ω, c)-periodic if

$$f(t+\omega) = cf(t), \quad \text{for all} \quad t \in I.$$

The number ω is said to be a c-period of $f(\cdot)$. The space of all (ω, c)-periodic functions $f : I \to E$ we be denoted with $P_{\omega,c}(I : E)$.

(ω, c)-almost periodic functions

Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. We say that a continuous function $f : I \to E$ is (ω, c)-almost periodic(uniformly recurrent, almost automorphic) if the function $f_{\omega,c}(t) = c^{-\frac{t}{\omega}} f(t)$, $t \in I$ is almost periodic (uniformly recurrent, almost automorphic).

Let $\omega > 0$, $c_1, \omega_1 > 0$, $c_2, \omega_2 > 0$, $f(\cdot)$ is (ω_1, c_1)-almost periodic and g is (ω_2, c_2)-almost periodic and both functions are scalar valued. Set $c = c_1^{\frac{\omega}{\omega_1}} c_2^{\frac{\omega}{\omega_2}}$. Then the function $fg(\cdot)$ is (ω, c)-almost periodic.

Let $f : \mathbb{R} \to E$. The function $f(\cdot)$ is (ω, c)-almost periodic if and only if the function $\check{f}(\cdot) = f(-\cdot)$ is $(\omega, \frac{1}{c})$-almost periodic function.

Let $E = \mathbb{C}$, $c \in \mathbb{C} \backslash \{0\}$, $\omega > 0$, $f : I \to \mathbb{C}$ and $\inf_{x \in I} |f(x)| > m > 0$. The following statements hold:

i) If $|c| = 1$ and $f(\cdot)$ is (ω, c)-almost periodic function, then the function $(1/f)(\cdot)$ is $(\omega, 1/c)$-almost periodic function.

ii) If $|c| \leq 1$, $I = [0, \infty)$ and $f(\cdot)$ is (ω, c)-almost periodic then the function $(1/f)(\cdot)$ is $(\omega, 1/c)$-almost periodic function.

Theorem 1

Suppose that $f : \mathbb{R} \to E$ satisfies the function $f_{\omega,c}(\cdot)$ is a bounded almost periodic and $c^{-\dot{\frac{}{\omega}}}\psi(\cdot) \in L^1(\mathbb{R})$. Then the function $c^{-\dot{\frac{}{\omega}}}(\psi * f)\ (\cdot)$ is bounded uniformly continuous and the function $(\psi * f)(\cdot)$ is (ω, c)-almost periodic.

Stepanov p-uniformly recurrent functions

Let $1 \leq p < \infty$. A function $f \in L^p_{loc}(I : E)$ is said to be Stepanov p-uniformly recurrent if the function $\hat{f} : I \to L^p([0,1] : E)$, defined by $\hat{f}(t)(s) = f(t+s)$, $t \in I$, $s \in [0,1]$ is uniformly recurrent.

Stepanov (p, ω, c)-almost periodic functions

Let $p \in [1, \infty)$, $c \in \mathbb{C} \backslash \{0\}$ and $\omega > 0$. It is said that a function $f \in L^p_{loc}(I : E)$ is Stepanov (p, ω, c)-almost periodic ((p, ω, c)-uniformly recurrent) function if the function $f_{\omega,c}(t) = c^{-\frac{t}{\omega}} f(t)$, $t \in I$ is Stepanov p-almost periodic (Stepanov p-uniformly recurrent function).

Theorem 2

Let $I = \mathbb{R}$ or $I = [0, \infty)$ and define the function $G : I \times X \to E$ by $G(t, f_{\omega,c}(t) = c_1^{-\frac{t}{\omega_1}} F(t, f(t))$, where $c_1 \in \mathbb{C}\backslash\{0\}$ and $\omega_1 > 0$, $t \in I$ and $x \in X$. Suppose that the following conditions hold:

i) The function $G : I \times X \to E$ is Stepanov p-uniformly recurrent, with $p > 1$ and there exists a number $r \geq \max(p, p/(p-1))$ and a function $L_G \in L_S^r(I)$ such that

$$\|G(t,x) - G(t,y)\| \leq L_G(t)\|x - y\|_X, \quad t \in I,\ x, y \in X.$$

ii) The function $f_{\omega,c} : I \to X$ is Stepanov p-uniformly recurrent and there exists a set $E \subseteq I$ with $m(E) = 0$ such that $K := \{f_{\omega,c}(t) \,:\, t \in I\backslash E\}$ is relatively compact in X.

iii) For every compact set $K \subseteq X$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\in I} \sup_{u\in K} \int_0^1 \|G(t+s+\alpha_n, u) - G(t+s, u)\|^p \, ds = 0$$

and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|\hat{f}_{\omega,c}(t+\alpha_n) - \hat{f}_{\omega,c}(t)\|_{L^p([0,1]:X)} = 0.$$

Then $q := \frac{pr}{p+r} \in [1, p)$ and $F(\cdot, f(\cdot))$ is Stepanov (q, ω_1, c_1)-uniformly recurrent.

We will examine the invariance of (ω, c)-almost periodic properties of the infinite convolution product

$$F(t) = \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

Theorem 3

Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t)) \subseteq L(E, X)$ is a strongly continuous operator family satisfying that

$$\sum_{k=0}^{\infty} \|e^{-\frac{\cdot}{\omega}} R(\cdot)\|_{L^q[k,k+1]} < \infty.$$

If $c^{-\frac{\cdot}{\omega}} f : \mathbb{R} \to X$ is S^p-almost periodic, then the function $F : \mathbb{R} \to X$ given before is well defined and (ω, c)-almost periodic.

Example 1

The unique solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, with the initial condition $u(x,0) = f(x)$ is given by

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\, ds, \quad x \in \mathbb{R}, \ \ t \geq 0.$$

If $e^{-\frac{\cdot}{\omega}} e^{-\frac{\cdot^2}{4t_0}} \in L^1(\mathbb{R})$ and f is (ω, c)-uniformly recurrent ((ω, c)-almost periodic) function then by Theorem 1, the solution $x \mapsto u(x, t_0)$, $x \in \mathbb{R}$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic).

Example 2

Let we consider the the following fractional Cauchy inclusion

$$D^{\gamma}_{t,+}u(t) \in \mathcal{A}u(t) + f(t), \quad t \in \mathbb{R},$$

where $D^{\gamma}_{t,+}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1]$, $f : \mathbb{R} \to E$ satisfies certain properties and $\mathcal{A}$ is a closed multivalued linear operator. Assuming certain technical conditions, using Theorem 3, if $c^{-\frac{\cdot}{\omega}} f : \mathbb{R} \to X$ is S^p-almost periodic, then the solutions of the upper fractional Cauchy inclusion are (ω, c)-almost periodic.

Example 3

Let we consider wave equation

$$u_{xx}(x,t) = u_{tt}(x,t), \quad x \in \mathbb{R}, t \geq 0$$

with initial conditions $u(x,0) = f(x)$, $x \in \mathbb{R}$, $u_t(x,0) = g(x)$, $x \in \mathbb{R}$. The solution is given with

$$u(x,t) = \frac{1}{2}\Big(f(x+t) + f(x-t)\Big) + \frac{1}{2}\int\limits_{x-t}^{x+t} g(s)\,ds, \;\; x \in \mathbb{R}, \; t \geq 0.$$

Let $|c| \leq 1$ and suppose that the function $g : [-t_0, \infty) \to \mathbb{C}$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic) function. Then the solution $u(x,t)$ of the wave equation is (ω, c)-uniformly recurrent ((ω, c)-almost periodic).

THANK YOU FOR YOUR ATTENTION